

## 邊界元素法應用於潮流引起擴散

在水平面內取直角座標系(x, y), z 軸垂直向上，潮流在 x, y 方向分量為 u, v。不可壓縮流的連續方程式如下

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(1)

利用 Navier-Stokes 方程式描述流的運動

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = X - \frac{1}{\rho} \frac{\partial P}{\partial x} + fv - \frac{1}{\rho} \frac{\partial F_x}{\partial z}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = Y - \frac{1}{\rho} \frac{\partial P}{\partial y} - fu - \frac{1}{\rho} \frac{\partial F_y}{\partial z}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = Z - \frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{1}{\rho} \frac{\partial F_z}{\partial z}$$

(4)

$\rho$  為海水密度，P 為壓力，f 柯氏力，X、Y、Z 為外力， $F_x$ 、 $F_y$ 、 $F_z$  為摩擦力。

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假定為淺水條件，且 X=Y=0，Z=-g，(4)式左邊的水粒子於 z 方向加速度與重力加速度相比為小，可忽略不計，即

$$0 = -g - \frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{1}{\rho} \frac{\partial F_x}{\partial z} \dots \dots \quad (5)$$

忽略垂直方向摩擦力變化時，上式可改寫為

$$0 = -g - \frac{1}{\rho} \frac{\partial P}{\partial z}$$

即

$$dP = -\rho g dz$$

由海底 ( $z = -h$ ) 積分到海面 ( $z = \zeta$ ，水位的上升量) 時，得

$$P = \int_{-h}^{\zeta} \rho g dz = P_0 - g \int_{-h}^{\zeta} \rho dz \dots \dots \quad (6)$$

$P_0$  為大氣壓力，x、y 軸方向的壓力分量為

$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} - g \frac{\partial}{\partial x} \int_{-h}^{\zeta} \rho dz$$

$$\frac{\partial P}{\partial y} = \frac{\partial P_0}{\partial y} - g \frac{\partial}{\partial y} \int_{-h}^{\zeta} \rho dz$$

假設壓力梯度變化小，得

$$\frac{\partial P_0}{\partial x} = \frac{\partial P_0}{\partial y} = 0$$

即

$$\frac{\partial P}{\partial x} = -g \frac{\partial}{\partial x} \int_{-h}^{\zeta} \rho dz$$

$$\frac{\partial P}{\partial y} = -g \frac{\partial}{\partial y} \int_{-h}^{\zeta} \rho dz$$



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因

$$g \frac{\partial}{\partial x} \int_{-h}^{\zeta} \rho dz = g \int_{-h}^{\zeta} \frac{\partial \rho}{\partial x} dz + g \rho \Big|_{-h}^{\zeta} \frac{\partial \zeta}{\partial x} - g \rho \Big|_{-h}^{\zeta} \frac{\partial (-h)}{\partial x}$$

假定海底地形變化緩和，可忽略  $\partial(-h)/\partial x$  及  $\partial(-h)/\partial y$ 。

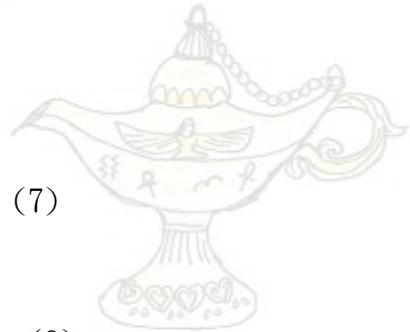
假定海水密度為定值，不隨水平方向變化，得

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = 0$$

故得

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial \zeta}{\partial x}$$

$$\frac{\partial P}{\partial y} = \rho g \frac{\partial \zeta}{\partial y}$$



(7)

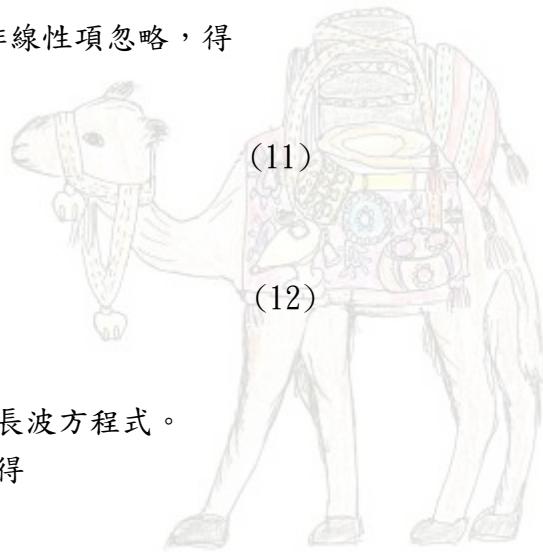
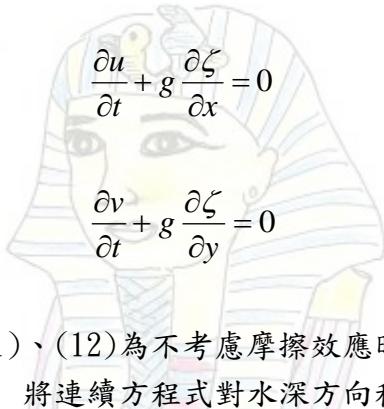
(8)

(2)、(3)式可以改寫為

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -g \frac{\partial \zeta}{\partial x} \quad (9)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -g \frac{\partial \zeta}{\partial y} \quad (10)$$

為了簡化計算，將(9)、(10)式的非線性項忽略，得



(11)、(12)為不考慮摩擦效應時的長波方程式。

將連續方程式對水深方向積分得

$$\int_{-h}^{\zeta} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] dz = - \int_{-h}^{\zeta} \frac{\partial w}{\partial z} dz = -w|_{\zeta} + w|_{-h}$$

即

$$\left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \int_{-h}^{\zeta} dz = -w|_{\zeta} + w|_{-h} \quad (13)$$

$$w|_{\zeta} = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \quad (14)$$

$$w|_{-h} = -\frac{dh}{dt} = -\frac{\partial h}{\partial t} - u \frac{\partial h}{\partial x} - v \frac{\partial h}{\partial y} \quad (15)$$

將(14)、(15)式代入(13)式得

$$\begin{aligned} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] (\zeta + h) &= -\frac{\partial \zeta}{\partial t} - \frac{\partial h}{\partial t} - u \left( \frac{\partial \zeta}{\partial x} + \frac{\partial h}{\partial x} \right) - v \left( \frac{\partial \zeta}{\partial y} + \frac{\partial h}{\partial y} \right) \\ \frac{\partial \zeta}{\partial t} + \frac{\partial h}{\partial t} + \left\{ \frac{\partial u}{\partial x} (\zeta + h) + u \left( \frac{\partial \zeta}{\partial x} + \frac{\partial h}{\partial x} \right) \right\} + \left\{ \frac{\partial v}{\partial y} (\zeta + h) + v \left( \frac{\partial \zeta}{\partial y} + \frac{\partial h}{\partial y} \right) \right\} &= 0 \end{aligned}$$

由於海底為固定底床，

$$\frac{\partial h}{\partial t} = 0$$

得

$$\frac{\partial \zeta}{\partial t} + \frac{\partial u(\zeta+h)}{\partial x} + \frac{\partial v(\zeta+h)}{\partial y} = 0 \quad (16)$$

將(11)式對 x 作偏微分，(12)式對 y 作偏微分得

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial x} \left( g \frac{\partial \zeta}{\partial x} \right) = \frac{\partial^2 u}{\partial x \partial t} + g \frac{\partial^2 \zeta}{\partial x^2} = 0 \quad (17)$$

$$\frac{\partial}{\partial y} \left( \frac{\partial v}{\partial t} + g \frac{\partial \zeta}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial t} \right) + \frac{\partial}{\partial y} \left( g \frac{\partial \zeta}{\partial y} \right) = \frac{\partial^2 v}{\partial y \partial t} + g \frac{\partial^2 \zeta}{\partial y^2} = 0 \quad (18)$$

將(16)式對時間作偏微分，代入(11)、(12)、(17)、(18)式的條件得

$$\frac{\partial}{\partial x} \left( h \frac{\partial \zeta}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial \zeta}{\partial y} \right) - \frac{1}{g} \frac{\partial^2 \zeta}{\partial t^2} = 0 \quad (19)$$

開放海域內無結構物存在時，潮汐引起的水位作  $\zeta_0 \exp(-i\sigma t)$  形式的簡諧運動， $\sigma$  為角週頻率 ( $= 2\pi/T$ ，T=週期)， $\zeta_0$  為水位振幅，水位為

$$\zeta_i(x, y; t) = -i\zeta_0 \exp \left\{ -i [k(x \cos \varpi + y \sin \varpi) + \sigma t] \right\}$$

海域受海底地形及結構物等影響引起水位變化  $\zeta(x, y)$  可以下式表示

$$\zeta(x, y) = w(x, y) + \zeta_i(x, y) \quad (20)$$

潮流方向為與 x 軸呈  $\varpi$  角度，週波數 k 可由下式求得

$$k^2 = \sigma^2 / gh$$

(19)式可改寫成

$$\frac{\partial}{\partial x} \left( h \frac{\partial \zeta}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial \zeta}{\partial y} \right) + k^2 h \zeta_i = 0$$

令 H=1-h，將(20)式代入上式可得

$$\nabla^2 w + k^2 w = H \nabla^2 w + \nabla H \cdot \nabla w - \alpha(x, y) \quad (21)$$

$$\alpha = \left( \frac{\partial}{\partial x} \left( h \frac{\partial \zeta_i}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial \zeta_i}{\partial y} \right) + k^2 h \zeta_i - k^2 w \right)$$

將(21)式乘以加權函數  $G$  並積分得

$$\int G(\nabla^2 w + k^2 w) d\Omega = \int GH \nabla^2 w d\Omega + \int G \nabla H \cdot \nabla w d\Omega - \int G \alpha(x, y) d\Omega$$

利用 Green 定理，上式可改寫成

$$\begin{aligned} \int w(\nabla^2 G + k^2 G) d\Omega &= \int w \nabla \cdot (H \nabla G) d\Omega \text{ 載滿珠寶的駱駝} \\ &+ \int GH \frac{\partial w}{\partial n} d\Gamma - \int Hw \frac{\partial G}{\partial n} d\Gamma + \int w \frac{\partial G}{\partial n} d\Gamma - \int G \frac{\partial w}{\partial n} d\Gamma - \int G \alpha(x, y) d\Omega \end{aligned}$$

上式左邊，加權函數  $G$  能滿足下式埃及尼羅河之旅

$$\nabla^2 G + k^2 G = -\delta(Q - P)$$

其基本解為

$$G = \frac{i}{4} H_0^{(1)}(kr)$$

式中， $H_0^{(1)}$  為第 1 類 0 次 Hankel 函數，故線性長波方程式的積分表示如下

$$\begin{aligned} \gamma w &= \int w \nabla \cdot (H \nabla G) d\Omega \\ &+ \int GH \frac{\partial w}{\partial n} d\Gamma - \int Hw \frac{\partial G}{\partial n} d\Gamma + \int w \frac{\partial G}{\partial n} d\Gamma - \int G \frac{\partial w}{\partial n} d\Gamma - \int G \alpha(x, y) d\Omega \end{aligned}$$

在邊界上  $\gamma = 0.5$ ，在領域內  $\gamma = 1$ 。

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回應用邊界元素法解析海洋擴散